

FRAMES AND OPERATORS IN SCHATTEN CLASSES

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ABSTRACT. Let T be a compact operator on a separable Hilbert space H . We show that, for $2 \leq p < \infty$, T belongs to the Schatten class S_p if and only if $\{\|Tf_n\|\} \in \ell^p$ for *every* frame $\{f_n\}$ in H ; and for $0 < p \leq 2$, T belongs to S_p if and only if $\{\|Tf_n\|\} \in \ell^p$ for *some* frame $\{f_n\}$ in H . Similar conditions are also obtained in terms of the sequence $\{\langle Tf_n, f_n \rangle\}$ and the double-indexed sequence $\{\langle Tf_n, f_m \rangle\}$.

1. INTRODUCTION

Let H be a separable Hilbert space. A sequence $\{f_n\} \subset H$ is called a *frame* for H if there exist positive constants C_1 and C_2 such that

$$C_1\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq C_2\|f\|^2$$

for all $f \in H$. The numbers C_1 and C_2 are certainly not unique. The optimal upper constant, $\inf C_2$, will be called the upper frame bound for $\{f_n\}$. Similarly, the optimal lower constant, $\sup C_1$, will be called the lower frame bound for $\{f_n\}$. A frame is called *tight* if its lower and upper frame bounds are the same. Also, a frame is called *Parseval* or *normalized tight* if its lower and upper frame bounds are both 1. See [3] for an introduction to the theory of frames.

The *singular values* or *s-numbers* of a compact operator T on H are the square roots of the positive eigenvalues of the operator T^*T , where T^* denotes the adjoint of T . Equivalently, this is the sequence of positive eigenvalues of $|T| = (T^*T)^{1/2}$. We always arrange the singular values of T , $\{\lambda_n\}$, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, with each eigenvalue of multiplicity k repeated k times in the sequence.

Given $0 < p < \infty$, the *Schatten p -class* of H , denoted $S_p(H)$ or simply S_p , is defined as the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to ℓ^p . It is well known that S_p is a two-sided ideal in the full algebra $\mathcal{L}(H)$ of all bounded linear operators on H . Also,

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when equipped with

$$\|T\|_p = \left[\sum_{n=1}^{\infty} \lambda_n^p \right]^{\frac{1}{p}},$$

S_p is a Banach space when $1 \leq p < \infty$ and a complete metric space when $0 < p < 1$. Two special cases are especially interesting in operator theory: S_1 is called the trace class and S_2 is the Hilbert-Schmidt class. See [4, 8, 9] for basic properties of Schatten classes.

Operators in Schatten classes can often be described by their action on orthonormal bases. For example, a positive operator $T \in \mathcal{L}(H)$ belongs to the trace class S_1 if and only if $\sum \langle T e_n, e_n \rangle < \infty$, where $\{e_n\}$ is any given orthonormal basis for H . Similarly, an operator $T \in \mathcal{L}(H)$ belongs to the Hilbert-Schmidt class S_2 if and only if $\sum \|T e_n\|^2 < \infty$, where $\{e_n\}$ is any given orthonormal basis for H . See [4, 8, 9] again for these and other related results.

It is clear that any orthonormal basis is a frame, with frame bounds equal to 1. The purpose of this article is to study Schatten class operators in terms of frames. We state our main results as follows.

Theorem A. *Suppose T is a compact operator on H and $2 \leq p < \infty$. Then the following conditions are equivalent.*

- (a) $T \in S_p$.
- (b) $\{\|T e_n\|\} \in \ell^p$ for every orthonormal basis $\{e_n\}$ in H .
- (c) $\{\|T f_n\|\} \in \ell^p$ for every frame $\{f_n\}$ in H .

Furthermore, we always have

$$\|T\|_p^p = \sup \sum_{n=1}^{\infty} \|T e_n\|^p = \sup \sum_{n=1}^{\infty} \|T f_n\|^p,$$

where the first supremum is taken over all orthonormal bases $\{e_n\}$ and the second supremum is taken over all frames $\{f_n\}$ with upper frame bound less than or equal to 1.

Theorem B. *Suppose T is a bounded operator on H and $0 < p \leq 2$. Then the following conditions are equivalent.*

- (a) $T \in S_p$.
- (b) $\{\|T e_n\|\} \in \ell^p$ for some orthonormal basis $\{e_n\}$ in H .
- (c) $\{\|T f_n\|\} \in \ell^p$ for some frame $\{f_n\}$ in H .

Furthermore, we always have

$$\|T\|_p^p = \inf \sum_{n=1}^{\infty} \|T e_n\|^p = \inf \sum_{n=1}^{\infty} \|f_n\|^{2-p} \|T f_n\|^p = \inf \sum_{n=1}^{\infty} \|T f_n\|^p,$$

where the first infimum is taken over all orthonormal bases, the second infimum is taken over all frames with lower frame bound greater than or equal to 1, and the third infimum is taken over all Parseval frames $\{f_n\}$.

The conditions above concerning orthonormal basis are more or less well known to experts in the field. But the necessary *and* sufficient conditions stated here do not seem to have appeared anywhere before. Partial statements in terms of orthonormal basis can be found in [4, 8, 9] for example. We will include the treatment for orthonormal bases in the paper for the sake of completeness.

It is interesting to observe the sharp contrast between the cases $p \geq 2$ and $p \leq 2$: in the first case sup is used to compute the norm $\|T\|_p$, while in the second case inf must be used. We will also construct examples to show that the cut-off at $p = 2$ is necessary, and the result at the cut-off value $p = 2$ is particularly nice.

In addition to Theorems A and B, we will obtain corresponding results in terms of the sequence $\{\langle T f_n, f_n \rangle\}$ and the double-indexed sequence $\{\langle T f_n, f_k \rangle\}$. But in these cases it is sometimes *necessary* to require additional assumptions on the operator T , such as T being positive or self-adjoint.

The relationship between frames and operators in Schatten classes has been studied by several authors in the past few years. See [1, 2, 6] and references therein. There is some overlap between the present paper and the papers just referenced. However, the approach here is different, the results here are *complete*, and the proofs here are simpler and more natural.

2. THE CASE WHEN p IS LARGE

The description of operators in the Schatten class S_p depends on the range of p . In this section we focus on the case when p large. We begin with the following lemma which is well known to experts. This is the only result from the theory of frames that we will use in the paper, so we include a short proof here for the reader's easy reference.

Lemma 1. *Suppose $\{e_n\}$ is an orthonormal basis and $\{f_n\}$ is a frame for H . Then the operator $A : H \rightarrow H$ defined by*

$$A \left(\sum_{k=1}^{\infty} c_k e_k \right) = \sum_{k=1}^{\infty} c_k f_k$$

is a well-defined bounded linear operator. Furthermore, $\|A\|^2$ is between the lower and upper frame bounds of $\{f_n\}$, and AA^ is invertible on H .*

Proof. If f is any vector in H , then

$$\begin{aligned} \left| \left\langle A \left(\sum_{k=1}^N c_k e_k \right), f \right\rangle \right|^2 &= \left| \sum_{k=1}^N c_k \langle f_k, f \rangle \right|^2 \\ &\leq \sum_{k=1}^N |c_k|^2 \sum_{k=1}^N |\langle f_k, f \rangle|^2 \\ &\leq C_2 \|f\|^2 \left\| \sum_{k=1}^N c_k e_k \right\|^2, \end{aligned}$$

where C_2 is the upper frame bound for $\{f_n\}$.

Therefore, by the Hahn-Banach theorem, A extends to a bounded linear operator on H with $\|A\|^2 \leq C_2$, namely,

$$A \left(\sum_{k=1}^{\infty} c_k e_k \right) = \sum_{k=1}^{\infty} c_k f_k,$$

where $\{c_k\} \in \ell^2$. If $\langle f, f_k \rangle = 0$ for all k , then it follows from the definition of frame that $f = 0$. Therefore, A has dense range.

For any vector $f \in H$, we have

$$A^* f = \sum_{n=1}^{\infty} \langle A^* f, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle f, A e_n \rangle e_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle e_n.$$

It follows that

$$\|A^* f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \geq C_1 \|f\|^2$$

for $f \in H$, where C_1 is the lower frame bound for $\{f_n\}$. This shows that $\|A\|^2 = \|A^*\|^2 \geq C_1$ and A^* is one-to-one and has closed range. Furthermore, for any $f \in H$, we have

$$C_1 \|f\|^2 \leq \|A^* f\|^2 = \langle A A^* f, f \rangle \leq \|A A^* f\| \|f\|.$$

It follows that $\|A A^* f\| \geq C_1 \|f\|$ for all $f \in H$, so that $A A^*$ is one-to-one and has closed range. Since $\text{ran}(A A^*)^\perp = \ker(A A^*) = (0)$, $A A^*$ must be onto. Therefore, $A A^*$ is invertible. \square

As a consequence of the invertibility of $A A^*$, we see that the operator A above is actually onto. Therefore, every vector $f \in H$ admits a representation of the form

$$f = \sum_{n=1}^{\infty} c_n f_n,$$

where $\{c_n\} \in \ell^2$. Note that A is generally not one-to-one. For example, a frame may contain a certain vector that is repeated a finite number of times. In this case, the associated operator A is obviously not one-to-one.

Theorem 2. *Suppose T is a compact operator on H and $2 \leq p < \infty$. Then the following statements are equivalent.*

- (a) T is in the Schatten class S_p .
- (b) $\{\|Te_n\|\} \in \ell^p$ for every orthonormal basis $\{e_n\}$ in H .
- (c) $\{\|Tf_n\|\} \in \ell^p$ for every frame $\{f_n\}$ in H .

Moreover, we always have

$$\|T\|_p^p = \sup \sum_{n=1}^{\infty} \|Te_n\|^p = \sup \sum_{n=1}^{\infty} \|Tf_n\|^p,$$

where the first supremum is taken over all orthonormal bases $\{e_n\}$ and the second supremum is taken over all frames $\{f_n\}$ with upper frame bound less than or equal to 1.

Proof. The equivalence of conditions (a) and (b) is well known. See Theorem 1.33 of [9] for example. Note that Theorem 1.33 of [9] was stated and proved in terms of orthonormal sets. Since every orthonormal set can be expanded to an orthonormal basis, the result remains true when the phrase “orthonormal sets” is replaced by “orthonormal bases”.

Since every orthonormal basis is a frame, it is trivial that condition (c) implies (b).

To prove that (a) and (b) together imply (c), we fix an orthonormal basis $\{e_n\}$ and a frame $\{f_n\}$ for H and consider the operator A defined in Lemma 1. If T is in S_p , then so is the operator $S = TA$. Apply condition (b) to the operator S , we obtain

$$\sum_{n=1}^{\infty} \|Tf_n\|^p = \sum_{n=1}^{\infty} \|TAe_n\|^p = \sum_{n=1}^{\infty} \|Se_n\|^p < \infty.$$

This completes the proof of the equivalence of conditions (a), (b), and (c).

The equality

$$\|T\|_p^p = \sup \sum_{n=1}^{\infty} \|Te_n\|^p$$

was established in Theorem 1.33 of [9]. Since every orthonormal basis is a frame with frame bounds equal to 1, we clearly have

$$\|T\|_p^p = \sup \sum_{n=1}^{\infty} \|Te_n\|^p \leq \sup \sum_{n=1}^{\infty} \|Tf_n\|^p.$$

This along with the arguments in the previous paragraph shows that

$$\sum_{n=1}^{\infty} \|Tf_n\|^p = \sum_{n=1}^{\infty} \|TAe_n\|^p \leq \|TA\|_p^p.$$

It is well known (see [4, 9] for example) that $\|TA\|_p \leq \|T\|_p \|A\|$, which combined with the estimate for $\|A\|$ in Lemma 1 shows that $\|TA\|_p \leq \|T\|_p$ whenever $\{f_n\}$ has upper frame bound less than or equal to 1. This shows that

$$\sup \sum_{n=1}^{\infty} \|Tf_n\|_p^p \leq \|T\|_p^p,$$

and completes the proof of the theorem. \square

It is possible to obtain a version of Theorem 2 without the a priori assumption that T be compact. This will be done using an approximation argument based on the following lemma.

Lemma 3. *Suppose T and T_k , $k \geq 1$, are bounded linear operators on H . If $1 < p < \infty$, $T_k \rightarrow T$ in the weak operator topology, and $\|T_k\|_p \leq C$ for some constant C and all $k \geq 1$, then $\|T\|_p \leq C$.*

Proof. Let S be a finite-rank operator and $\{e_n\}$ be an orthonormal basis of H such that $TS(e_n) = 0$ for all but a finite number of n . Then

$$\text{Tr}(TS) = \sum_{n=1}^{\infty} \langle TS e_n, e_n \rangle = \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} \langle T_k S e_n, e_n \rangle = \lim_{k \rightarrow \infty} \text{Tr}(T_k S).$$

Since the Banach dual of S_p is S_q , $1/p + 1/q = 1$, under the pairing induced by the trace, we have

$$|\text{Tr}(T_k S)| \leq \|T_k\|_p \|S\|_q \leq C \|S\|_q, \quad k \geq 1.$$

It follows that $|\text{Tr}(TS)| \leq C \|S\|_q$ for all finite rank operators S . Since the set of finite rank operators is dense in S_q , we conclude that $T \in S_p$ and $\|T\|_p \leq C$. \square

Theorem 4. *When $2 \leq p < \infty$, the following conditions are equivalent for any bounded linear operator T on H :*

- (i) $T \in S_p$.
- (ii) *There exists a constant $C > 0$ such that*

$$\sum_{n=1}^{\infty} \|Te_n\|^p \leq C$$

for every orthonormal basis $\{e_n\}$.

(iii) *There exists a constant $C > 0$ such that*

$$\sum_{n=1}^{\infty} \|Tf_n\|^p \leq C$$

for every frame $\{f_n\}$ with upper frame bound no greater than 1.

Proof. All we have to show here is that condition (ii) implies (i) without the a priori assumption that T be compact. This can be done with the help of an approximation argument. More specifically, we fix an increasing sequence $\{P_k\}$ of finite-rank projections such that $\{P_k\}$ converges to the identity operator in the strong operator topology and let $T_k = P_k T$ for $k \geq 1$. Each T_k is a finite-rank operator, so condition (ii) along with Theorem 2 gives

$$\|T_k\|_p^p = \sup \sum_{n=1}^{\infty} \|T_k e_n\|^p \leq \sup \sum_{n=1}^{\infty} \|T e_n\|^p \leq C$$

for all $k \geq 1$, where the suprema are taken over all orthonormal bases $\{e_n\}$. Since $T_k \rightarrow T$ in the strong operator topology, it follows from Lemma 3 that $\|T\|_p^p \leq C$. \square

It is natural to ask the following question: suppose $p \geq 2$ and $\{\|Tf_n\|\} \in \ell^p$ for *some* frame $\{f_n\}$, does it imply that $T \in S_p$? The answer is yes for $p = 2$ but no for $p > 2$. We will get back to the case $p = 2$ in Section 4 but will now settle the case $p > 2$.

Proposition 5. *Suppose $2 < p < \infty$, $\varepsilon > 0$ (not necessarily small), and T is any operator in $S_{p+\varepsilon} - S_p$. Then there exists a frame $\{f_n\}$ for H such that $\{\|Tf_n\|\} \in \ell^p$.*

Proof. Suppose that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n$$

is the canonical decomposition of T , where $\{\lambda_n\}$ is the singular value sequence of T which is arranged in nonincreasing order and repeated according to multiplicity. Thus we have $\{\lambda_n\} \in \ell^{p+\varepsilon}$ but $\{\lambda_n\} \notin \ell^p$.

Let $\{e'_n\}$ denote an orthonormal basis for $\ker(T)$. Then $\{e_n\} \cup \{e'_n\}$ is an orthonormal basis for H . In fact, for any vector $x \in H$, we have $\langle x, e_n \rangle = 0$ for every n if and only if $Tx = 0$. Therefore, $\{e_n\}^\perp = \{e'_n\}$.

For every $n \geq 1$ choose a positive number δ_n such that $\delta_n^{p-2} = \lambda_n^\varepsilon$. Since $p > 2$, we have $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, so we can choose a sequence $\{N_n\}$ of positive integers such that $N_n \delta_n^2 \sim 1$ as $n \rightarrow \infty$. In other words, there exist positive constants c and C such that $c \leq N_n \delta_n^2 \leq C$ for all $n \geq 1$. Let $\{f_n\}$ be the sequence consisting of all vectors in $\{e'_n\}$, plus N_1 copies of the vector $\delta_1 e_1$, plus N_2 copies of the vector $\delta_2 e_2$, and so on.

For any vector $f \in H$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 &= \sum_{n=1}^{\infty} N_n |\langle f, \delta_n e_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle f, e'_n \rangle|^2 \\
&= \sum_{n=1}^{\infty} N_n \delta_n^2 |\langle f, e_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle f, e'_n \rangle|^2 \\
&\sim \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle f, e'_n \rangle|^2 \\
&= \|f\|^2.
\end{aligned}$$

This shows that $\{f_n\}$ is a frame for H .

On the other hand,

$$\begin{aligned}
\sum_{n=1}^{\infty} \|T f_n\|^p &= \sum_{n=1}^{\infty} N_n \|T(\delta_n e_n)\|^p = \sum_{n=1}^{\infty} N_n \delta_n^p \|T e_n\|^p \\
&= \sum_{n=1}^{\infty} N_n \delta_n^p \lambda_n^p = \sum_{n=1}^{\infty} N_n \delta_n^2 \delta_n^{p-2} \lambda_n^p \\
&\sim \sum_{n=1}^{\infty} \delta_n^{p-2} \lambda_n^p = \sum_{n=1}^{\infty} \lambda_n^{p+\varepsilon} < \infty.
\end{aligned}$$

This completes the proof of the proposition. \square

We also derive a version of the above proposition in terms of orthonormal bases.

Proposition 6. *Suppose $2 < p < \infty$, $\varepsilon > 0$, and $\{e_n\}$ is any orthonormal basis for H . Then there exists an operator $S \in S_{p+\varepsilon} - S_p$ such that $\{\|S e_n\|\} \in \ell^p$.*

Proof. Fix any operator $T \in S_{p+\varepsilon} - S_p$ and use Proposition 5 to select a frame $\{f_n\}$ such that $\{\|T f_n\|\} \in \ell^p$. Let A be the operator on H defined by $A e_n = f_n$, $n \geq 1$. By Lemma 1, the operator A is bounded and the operator AA^* is invertible. Let $S = TA$. Then $S \in S_{p+\varepsilon}$ because $S_{p+\varepsilon}$ is a two-sided ideal in the full operator algebra $\mathcal{L}(H)$. Since AA^* is invertible, we have $S \notin S_p$ as well. Otherwise, the operator $T(AA^*) = SA^*$ would be in S_p . Multiplying from the right by $(AA^*)^{-1}$ and using the fact that S_p is a two-sided ideal in $\mathcal{L}(H)$ again, we would then obtain that T is in S_p , a contradiction. Therefore, $S \in S_{p+\varepsilon} - S_p$ and

$$\sum_{n=1}^{\infty} \|S e_n\|^p = \sum_{n=1}^{\infty} \|T A e_n\|^p = \sum_{n=1}^{\infty} \|T f_n\|^p < \infty.$$

This completes the proof of the proposition. \square

Characterizations of Schatten classes can also be given in terms of the sequence $\{\langle Tf_n, f_n \rangle\}$ and the double-indexed sequence $\{\langle Tf_n, f_k \rangle\}$. We now proceed to the characterization of S_p based on the sequence $\{\langle Tf_n, f_n \rangle\}$.

Theorem 7. *Suppose $1 \leq p < \infty$ and S is a compact operator on H . Then the following conditions are equivalent.*

- (a) S belongs to S_p .
- (b) $\{\langle Se_n, e_n \rangle\} \in \ell^p$ for every orthonormal basis $\{e_n\}$ in H .
- (c) $\{\langle Sf_n, f_n \rangle\} \in \ell^p$ for every frame $\{f_n\}$ in H .

Furthermore, if S is self-adjoint, then

$$\|S\|_p^p = \sup \sum_{n=1}^{\infty} |\langle Se_n, e_n \rangle|^p = \sup \sum_{n=1}^{\infty} |\langle Sf_n, f_n \rangle|^p,$$

where the first supremum is taken over all orthonormal bases and the second supremum is taken over all frames with upper frame bound less than or equal to 1.

Proof. The equivalence of (a) and (b) follows from Theorem 1.27 in [9]. Note again that Theorem 1.27 in [9] is stated in terms of orthonormal sets. Since every orthonormal set can be expanded to an orthonormal basis, we see that Theorem 1.27 in [9] remains valid when the phrase “orthonormal sets” is replaced by “orthonormal bases”.

It is trivial that (c) implies (b).

To prove that (a) implies (c), first assume that S is positive. In this case, we can write $S = T^*T$, where T is the square root of S . Then $\langle Sf_n, f_n \rangle = \|Tf_n\|^2$ and the desired result follows from Theorem 2 and the fact that $S \in S_p$ if and only if $T \in S_{2p}$. When S is not necessarily positive, it is well known that we can write

$$S = (S_1 - S_2) + i(S_3 - S_4),$$

where each S_k is positive and belongs to S_p . By the already proved case for positive operators, $\{\langle S_k f_n, f_n \rangle\} \in \ell^p$ for each $1 \leq k \leq 4$. It follows that $\{\langle Sf_n, f_n \rangle\} \in \ell^p$.

It follows from the canonical decomposition for self-adjoint compact operators and the fact that every orthonormal basis is a frame with frame bounds 1 that

$$\|S\|_p^p \leq \sup \sum_{n=1}^{\infty} |\langle Se_n, e_n \rangle|^p \leq \sup \sum_{n=1}^{\infty} |\langle Sf_n, f_n \rangle|^p.$$

If $\{f_n\}$ is a frame with upper frame bound 1, then by the norm estimate for A in Lemma 1 we have $\|f_n\| \leq 1$ for every n , which together with

Theorem 2 gives

$$\sum_{n=1}^{\infty} |\langle Sf_n, f_n \rangle|^p \leq \sum_{n=1}^{\infty} \|Sf_n\|^p \leq \|S\|_p^p.$$

This shows that

$$\sup \sum_{n=1}^{\infty} |\langle Sf_n, f_n \rangle|^p \leq \|S\|_p^p,$$

and completes the proof of the theorem. \square

Note that the second assertion in Theorem 7 concerning the norm of S in S_p is false for operators that are not necessarily self-adjoint. A counter-example can be found on page 22 of [9]. Nevertheless, using the fact that every operator T admits a canonical decomposition $T = T_1 + iT_2$ with

$$|\langle Tf, f \rangle|^2 = |\langle T_1 f, f \rangle|^2 + |\langle T_2 f, f \rangle|^2,$$

where

$$T_1 = \frac{T + T^*}{2}, \quad T_2 = \frac{T - T^*}{2i},$$

are self-adjoint, we easily show that there still exists a positive constant C such that

$$C^{-1} \|T\|_p^p \leq \sup \sum_{n=1}^{\infty} |\langle Te_n, e_n \rangle|^p \leq C \|T\|_p^p$$

for all operators $T \in S_p$, where the supremum is taken over all orthonormal bases $\{e_n\}$. See the second part of the proof of Theorem 1.27 in [9]. Similarly, we have

$$C^{-1} \|T\|_p^p \leq \sup \sum_{n=1}^{\infty} |\langle Tf_n, f_n \rangle|^p \leq C \|T\|_p^p$$

for all operators $T \in S_p$, where the supremum is taken over all frames $\{f_n\}$ with upper frame bound less than or equal to 1.

If we remove the a priori assumption that T be compact, we obtain the following slightly different version of Theorem 7.

Theorem 8. *If $1 \leq p < \infty$ and S is a bounded linear operator on H , then the following conditions are equivalent:*

- (i) $S \in S_p$.
- (ii) *There exists a positive constant C such that*

$$\sum_{n=1}^{\infty} |\langle Se_n, e_n \rangle|^p \leq C$$

for every orthonormal basis $\{e_n\}$.

(iii) *There exists a positive constant C such that*

$$\sum_{n=1}^{\infty} |\langle Sf_n, f_n \rangle|^p \leq C$$

for every frame $\{f_n\}$ with upper frame bound less than or equal to 1.

Proof. This follows from Theorem 7, the remarks immediately following Theorem 7, and the same approximation argument used in the proof of Theorem 4. \square

Proposition 9. *If $1 < p < \infty$, $\varepsilon > 0$, and $S \in S_{p+\varepsilon} - S_p$ is positive, then there exists some frame $\{f_n\}$ such that $\{\langle Sf_n, f_n \rangle\} \in \ell^p$.*

Proof. Write $S = T^*T$, where $T = \sqrt{S}$. Then $T \in S_{2p+2\varepsilon} - S_{2p}$ and $\|Tf_n\|^{2p} = \langle Sf_n, f_n \rangle^p$. The desired result then follows from Proposition 5. \square

Proposition 10. *If $1 < p < \infty$, $\varepsilon > 0$, and $\{e_n\}$ is an orthonormal basis for H , then there exists a positive operator $S \in S_{p+\varepsilon} - S_p$ such that $\{\langle Se_n, e_n \rangle\} \in \ell^p$.*

Proof. By Proposition 6, there exists an operator $T \in S_{2p+2\varepsilon} - S_{2p}$ such that $\{\|Te_n\|\} \in \ell^{2p}$. Let $S = T^*T$. Then $S \in S_{p+\varepsilon} - S_p$ and the sequence $\langle Se_n, e_n \rangle = \|Te_n\|^2$ belongs to ℓ^p . \square

Next we proceed to the characterization of operators in Schatten classes in terms of the double-indexed sequence $\{\langle Tf_n, f_k \rangle\}$. We need the following lemma.

Lemma 11. *For any frame $\{f_n\}$ in H there exist positive constants c and C with the following properties.*

(a) *If $2 \leq p < \infty$, then*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \leq C \sum_{n=1}^{\infty} \|Tf_n\|^p$$

for all bounded linear operators T on H .

(b) *If $0 < p \leq 2$, then*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \geq c \sum_{n=1}^{\infty} \|Tf_n\|^p$$

for all bounded linear operators on H .

Proof. The desired estimates follow from Hölder's inequality and the definition of frames.

For $2 \leq p < \infty$, we have $0 < 2/p \leq 1$, so

$$\left[\sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \right]^{\frac{2}{p}} \leq \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^2 \leq C_1 \|Tf_n\|^2,$$

where C_1 is upper frame bound for $\{f_n\}$. It follows that for $C = C_1^{p/2}$ we have

$$\sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \leq C \|Tf_n\|^p, \quad n \geq 1,$$

so that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \leq C \sum_{n=1}^{\infty} \|Tf_n\|^p.$$

Similarly, for $0 < p \leq 2$, we have $0 < p/2 \leq 1$, so

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (|\langle Tf_n, f_k \rangle|^2)^{\frac{p}{2}} \\ &\geq \sum_{n=1}^{\infty} \left[\sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^2 \right]^{\frac{p}{2}} \\ &\geq C_2^{p/2} \sum_{n=1}^{\infty} [\|Tf_n\|^2]^{\frac{p}{2}} \\ &= c \sum_{n=1}^{\infty} \|Tf_n\|^p, \end{aligned}$$

where C_2 is the lower frame bound for $\{f_n\}$ and $c = C_2^{p/2}$. \square

It is clear that if $\{f_n\}$ happens to be an orthonormal basis, then both C and c can be taken to be 1 in Lemma 11.

Theorem 12. Suppose T is a compact operator on H and $2 \leq p < \infty$. Then the following conditions are equivalent.

- (a) $T \in S_p$.
- (b) The condition

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Te_n, e_k \rangle|^p < \infty$$

for every orthonormal basis $\{e_n\}$ in H .

(c) *The condition*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T f_n, f_k \rangle|^p < \infty$$

holds for every frame $\{f_n\}$ in H .

Furthermore, there exists a positive constant c such that

$$c \|T\|_p^p \leq \sup \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p \leq \sup \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T f_n, f_k \rangle|^p \leq \|T\|_p^p,$$

where the first supremum is taken over all orthonormal bases $\{e_n\}$ and the second supremum is taken over all frames $\{f_n\}$ with upper frame bound less than or equal to 1.

Proof. That condition (a) implies (c) follows from Theorem 2 and part (a) of Lemma 11. Since every orthonormal basis is a frame, it is trivial that condition (c) implies (b).

It remains to show that condition (b) implies (a). So we assume that condition (b) holds for an operator T . It is clear that condition (b) also holds for T^* , which implies that condition (b) holds for $T + T^*$ and $T - T^*$ as well. Write $T = T_1 + iT_2$, where $T_1 = (T + T^*)/2$ and $T_2 = (T - T^*)/(2i)$, and apply condition (b) to the self-adjoint operators T_1 and T_2 , we may as well assume that T is already self-adjoint.

But if T is self-adjoint, its canonical decomposition takes the form

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n,$$

where $\{\lambda_n\}$ is the singular value sequence of T and $\{e_n\}$ is an orthonormal set. If $\{\sigma_n\}$ is an orthonormal basis for $\ker(T)$, then $\{e'_n\} = \{e_n\} \cup \{\sigma_n\}$ is an orthonormal basis for H . Therefore, it follows from condition (b) and the relation $\text{ran}(T)^\perp = \text{ran}(T^*)^\perp = \ker(T)$ that

$$\sum_{n=1}^{\infty} \lambda_n^p = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e'_n, e'_k \rangle|^p < \infty.$$

The first norm estimate follows from the decomposition $T = T_1 + iT_2$ of T into a linear combination of self-adjoint operators and the canonical decomposition of self-adjoint compact operators. The second norm estimate is trivial. The third norm estimate follows from Theorem 2 and part (a) of Lemma 11. \square

Note that if T is self-adjoint, then the proof above actually shows

$$\|T\|_p^p = \sup \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p = \sup \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T f_n, f_k \rangle|^p.$$

We are not sure if this holds for general operators as well.

Once again, if we do not make the a priori assumption that T be compact, then Theorem 12 should be modified as follows.

Theorem 13. *For $2 \leq p < \infty$ and any bounded linear operator T the following conditions are equivalent:*

- (i) $T \in S_p$.
- (ii) *There exists a positive constant C such that*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p \leq C$$

for every orthonormal basis $\{e_n\}$.

- (iii) *There exists a positive constant C such that*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T f_n, f_k \rangle|^p \leq C$$

for every frame $\{f_n\}$ with upper frame bound less than or equal to 1.

Proof. This follows from Theorem 12 and the approximation argument used in the proof of Theorems 4 and 8. \square

Proposition 14. *Let $2 < p < \infty$, $\varepsilon > 0$, and $T \in S_{p+\varepsilon} - S_p$. There exists a frame $\{f_n\}$ such that*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle T f_n, f_k \rangle|^p < \infty.$$

Proof. This follows from Proposition 5 and part (a) of Lemma 11. \square

Proposition 15. *Suppose $2 < p < \infty$, $\varepsilon > 0$, and $\{e_n\}$ is an orthonormal basis for H . Then there exists an operator $T \in S_{p+\varepsilon} - S_p$ such that*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle T e_n, e_k \rangle|^p < \infty.$$

Proof. This follows from Proposition 6 and part (a) of Lemma 11. \square

3. THE CASE WHEN p IS SMALL

We begin this section with a simple example to show that the characterizations for operators in Schatten classes S_p obtained in the previous section for $2 \leq p < \infty$ are not true for the range $0 < p < 2$.

Fix any orthonormal basis $\{e_n\}$ and consider the vector

$$h = \sum_{n=1}^{\infty} \frac{e_n}{\sqrt{n} \log(n+1)}$$

in H . Define a rank one operator T on H by $Tx = \langle x, h \rangle h$. We have

$$Te_n = \langle e_n, h \rangle h = \frac{h}{\sqrt{n} \log(n+1)}, \quad n \geq 1.$$

It follows that

$$\sum_{n=1}^{\infty} \|Te_n\|^p = \|h\|^p \sum_{n=1}^{\infty} \frac{1}{[\sqrt{n} \log(n+1)]^p} = \infty$$

for any $0 < p < 2$. This shows that the characterizations obtained in Theorem 2 are no longer true for *any* $0 < p < 2$.

Later in this section we will actually show that for *any* operator $T \in S_p$, $0 < p < 2$, there exists a frame $\{f_n\}$ such that $\{\|Tf_n\|\} \notin \ell^p$. Nevertheless, there is still a nice characterization for operators in S_p , $0 < p \leq 2$, in terms of orthonormal bases and frames.

Theorem 16. *Suppose T is a positive operator on H and $0 < p \leq 1$. Then the following conditions are equivalent.*

- (a) $T \in S_p$.
- (b) $\{\langle Te_n, e_n \rangle\} \in \ell^p$ for some orthonormal basis $\{e_n\}$ in H .
- (c) $\{\langle Tf_n, f_n \rangle\} \in \ell^p$ for some frame $\{f_n\}$ in H .

Furthermore, we have

$$\begin{aligned} \|T\|_p^p &= \inf \sum_{n=1}^{\infty} |\langle Te_n, e_n \rangle|^p \\ &= \inf \sum_{n=1}^{\infty} \|f_n\|^{2(1-p)} \langle Tf_n, f_n \rangle^p \\ &= \inf \sum_{n=1}^{\infty} \langle Tf_n, f_n \rangle^p, \end{aligned}$$

where the first infimum is taken over all orthonormal bases, the second infimum is taken over all frames with lower frame bound greater than or equal to 1, and the third infimum is taken over all Parseval frames.

Proof. If T is positive and in S_p , then its canonical decomposition takes the form

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \sigma_n \rangle \sigma_n,$$

where $\{\lambda_n\} \in \ell^p$ is the singular value sequence of T and $\{\sigma_n\}$ is an orthonormal set in H . Since each λ_n is positive, we have $Tx = 0$ if and only if $\langle x, \sigma_n \rangle = 0$ for every n . Therefore, $\ker(T) = \{\sigma_n\}^\perp$. If $\{\sigma'_n\}$ is an orthonormal basis for $\ker(T)$, then $\{e_n\} = \{\sigma_n\} \cup \{\sigma'_n\}$ is an orthonormal basis for H . Since $T(\sigma_n) = \lambda_n \sigma_n$ for every n and $\{\lambda_n\} \in \ell^p$, we have

$$\sum_{n=1}^{\infty} |\langle Te_n, e_n \rangle|^p = \sum_{n=1}^{\infty} |\langle T\sigma_n, \sigma_n \rangle|^p = \sum_{n=1}^{\infty} \lambda_n^p = \|T\|_p^p < \infty.$$

This shows that condition (a) implies (b).

To prove that condition (b) implies (a), we use Theorem 1.26 and part (b) of Proposition 1.31 in [9]. More specifically,

$$\|T\|_p^p = \|T^p\|_1^p = \sum_{n=1}^{\infty} \langle T^p e_n, e_n \rangle \leq \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle^p < \infty$$

whenever $\{\langle T e_n, e_n \rangle\} \in \ell^p$. This shows that condition (b) implies (a), so conditions (a) and (b) are equivalent for any positive operator on H and

$$\|T\|_p^p = \inf \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle^p,$$

where the infimum is taken over all orthonormal bases $\{e_n\}$.

Since every orthonormal basis is a frame with both upper and lower frame bounds equal to 1, it is trivial that condition (b) implies (c), and

$$\inf \sum_{n=1}^{\infty} \|f_n\|^{2(1-p)} \langle T f_n, f_n \rangle^p \leq \inf \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle^p,$$

where the first infimum is taken over all frames $\{f_n\}$ with lower frame bound at least 1 and the second infimum is taken over all orthonormal bases $\{e_n\}$.

Finally, we assume that $\{\langle T f_n, f_n \rangle\} \in \ell^p$ for some frame $\{f_n\}$. Fix any orthonormal basis $\{e_n\}$, let A be the operator defined in Lemma 1, and set $S = A^* T A$. Then S is positive again and

$$\langle S e_n, e_n \rangle = \langle A^* T A e_n, e_n \rangle = \langle T A e_n, A e_n \rangle = \langle T f_n, f_n \rangle.$$

Thus $\{\langle S e_n, e_n \rangle\} \in \ell^p$. By the equivalence of (a) and (b), S is in S_p . Since S_p is a two-sided ideal in $\mathcal{L}(H)$, the operator $(A A^*) T (A A^*) = A S A^*$ also belongs to S_p . Multiply from both sides by $(A A^*)^{-1}$ (see Lemma 1), we

conclude that T is in S_p as well. This shows that condition (c) implies (a), and completes the proof of the equivalence of (a), (b), and (c).

We now proceed to prove that

$$\|T\|_p^p \leq \sum_{n=1}^{\infty} \|f_n\|^{2(1-p)} \langle T f_n, f_n \rangle^p$$

whenever $\{f_n\}$ is a frame with lower frame bound greater than or equal to 1. To this end, we fix such a frame and, without loss of generality, assume that the right-hand side above is finite (otherwise, the desired inequality is trivial) and $f_n \neq 0$ for each n . By part (b) of Proposition 1.31 in [9], we have

$$\begin{aligned} +\infty &> \sum_{n=1}^{\infty} \|f_n\|^{2(1-p)} \langle T f_n, f_n \rangle^p \\ &= \sum_{n=1}^{\infty} \|f_n\|^2 \left\langle T \frac{f_n}{\|f_n\|}, \frac{f_n}{\|f_n\|} \right\rangle^p \\ &\geq \sum_{n=1}^{\infty} \|f_n\|^2 \left\langle T^p \frac{f_n}{\|f_n\|}, \frac{f_n}{\|f_n\|} \right\rangle \\ &= \sum_{n=1}^{\infty} \langle T^p f_n, f_n \rangle. \end{aligned}$$

By the equivalence of (a) and (c), the operator T^p is in the trace class. If

$$T^p x = \sum_{k=1}^{\infty} \mu_k \langle x, \sigma_k \rangle \sigma_k$$

is the canonical decomposition for T^p , then

$$\langle T^p f_n, f_n \rangle = \sum_{k=1}^{\infty} \mu_k |\langle f_n, \sigma_k \rangle|^2$$

for every n . Since the lower frame bound of $\{f_n\}$ is greater than or equal to 1, it follows from Fubini's theorem and Theorem 1.26 in [9] that

$$\begin{aligned} \sum_{n=1}^{\infty} \langle T^p f_n, f_n \rangle^p &= \sum_{k=1}^{\infty} \mu_k \sum_{n=1}^{\infty} |\langle f_n, \sigma_k \rangle|^2 \\ &\geq \sum_{k=1}^{\infty} \mu_k \|\sigma_k\|^2 = \sum_{k=1}^{\infty} \mu_k \\ &= \|T^p\|_1 = \|T\|_p^p. \end{aligned}$$

This completes the proof that

$$\|T\|_p^p = \inf \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle^p = \inf \sum_{n=1}^{\infty} \|f_n\|^{2(1-p)} \langle T f_n, f_n \rangle^p,$$

where the first infimum is taken over all orthonormal bases and the second infimum is taken over all frames with lower frame bound at least 1.

Finally, if $\{f_n\}$ is a Parseval frame, then by the norm estimate for A in Lemma 1, we have $\|f_n\| \leq 1$ for every n . It follows that

$$\sum_{n=1}^{\infty} \|f_n\|^{2(1-p)} \langle T f_n, f_n \rangle^p \leq \sum_{n=1}^{\infty} \langle T f_n, f_n \rangle^p.$$

Combining this with the fact that every orthonormal basis is a Parseval frame, we obtain

$$\|T\|_p^p = \inf \sum_{n=1}^{\infty} \langle T f_n, f_n \rangle^p,$$

where the infimum is taken over all Parseval frames $\{f_n\}$. \square

Note that, unlike Theorem 7, we need to make the additional assumption that T be positive here. Without any extra assumption, Theorem 16 will be false. For example, if $\{e_n\}$ is any fixed orthonormal basis for H and T is the unilateral shift operator defined by $T(e_n) = e_{n+1}$, $n \geq 1$. Then it is clear that $\{\langle T e_n, e_n \rangle\} \in \ell^p$ for any $p > 0$, but T is not even compact.

As a consequence of Theorem 16 we obtain the following.

Theorem 17. *Suppose T is a bounded linear operator on H and $0 < p \leq 2$. Then the following conditions are equivalent.*

- (a) $T \in S_p$.
- (b) $\{\|T e_n\|\} \in \ell^p$ for some orthonormal basis $\{e_n\}$ in H .
- (c) $\{\|T f_n\|\} \in \ell^p$ for some frame $\{f_n\}$ in H .

Furthermore, we have

$$\|T\|_p^p = \inf \sum_{n=1}^{\infty} \|T e_n\|^p = \inf \sum_{n=1}^{\infty} \|f_n\|^{2-p} \|T f_n\|^p = \inf \sum_{n=1}^{\infty} \|T f_n\|^p,$$

where the first infimum is taken over all orthonormal bases, the second infimum is taken over all frames with lower frame bound greater than or equal to 1, and the third infimum is taken over all Parseval frames.

Proof. Consider $S = T^*T$ and apply Theorem 16 to the operator S . The desired result then follows from the identity $\langle S f_n, f_n \rangle = \|T f_n\|^2$ and the fact that $T \in S_p$ if and only if $S \in S_{p/2}$. \square

Theorem 18. *Suppose $0 < p \leq 2$ and T is a self-adjoint operator on H . Then the following conditions are equivalent.*

- (a) $T \in S_p$.
 (b) *There exists some orthonormal basis $\{e_n\}$ in H such that*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle T e_n, e_k \rangle|^p < \infty.$$

- (c) *There exists some frame $\{f_n\}$ in H such that*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle T f_n, f_k \rangle|^p < \infty.$$

Moreover, we have

$$\begin{aligned} \|T\|_p^p &= \inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p \\ &= \inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \|f_n\|^{2-p} |\langle T f_n, f_k \rangle|^p \\ &= \inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T f_n, f_k \rangle|^p, \end{aligned}$$

where the first infimum is taken over all orthonormal bases, the second infimum is taken over all frames with lower frame bound at least 1, and the third infimum is taken over all Parseval frames.

Proof. If $T \in S_p$ is self-adjoint, then there exists an orthonormal set $\{\sigma_n\}$ such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \sigma_n \rangle \sigma_n$$

for all $x \in H$, where $\{\lambda_n\} \in \ell^p$ is the nonzero eigenvalue sequence of T . Since each λ_n is nonzero, we see that $Tx = 0$ if and only if $x \perp \sigma_n$ for every n . Therefore, if $\{\sigma'_n\}$ is an orthonormal basis for $\ker(T)$, then $\{e_n\} =: \{\sigma_n\} \cup \{\sigma'_n\}$ is an orthonormal basis for H . Moreover,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T \sigma_n, \sigma_k \rangle|^p = \sum_{n=1}^{\infty} |\lambda_n|^p < \infty.$$

This shows that condition (a) implies (b).

Since every orthonormal basis is a frame, it is trivial that condition (b) implies (c). That (c) implies (a) follows from Theorem 17 and part (b) of Lemma 11.

If $\{f_n\}$ is a Parseval frame, then by Theorem 17 and the proof for part (b) of Lemma 11,

$$\|T\|_p^p \leq \sum_{n=1}^{\infty} \|Tf_n\|^p \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p.$$

It follows that

$$\|T\|_p^p \leq \inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p.$$

Since every orthonormal basis is a Parseval frame, we clearly have

$$\inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \leq \inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Te_n, e_k \rangle|^p.$$

The inequality

$$\inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Te_n, e_k \rangle|^p \leq \|T\|_p^p$$

follows from the first paragraph of this proof.

It follows from the proof of Lemma 11 that we always have

$$\sum_{n=1}^{\infty} \|f_n\|^{2-p} \sum_{k=1}^{\infty} |\langle Tf_n, f_k \rangle|^p \geq \sum_{n=1}^{\infty} \|f_n\|^{2-p} \|Tf_n\|^p,$$

where $0 < p \leq 2$ and $\{f_n\}$ is any frame with lower frame bound at least 1. Combining this with Theorem 17, we see that

$$\|T\|_p^p = \inf \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \|f_n\|^{2-p} |\langle Tf_n, f_k \rangle|^p,$$

where the infimum is taken over all frames with lower frame bound greater than or equal to 1. \square

We are not sure if the additional assumption that T be self-adjoint is necessary in Theorem 18 above. But we can show by an example that its proof will definitely not work if no additional assumption is placed on T . To see this, fix any orthonormal basis $\{e_n\}$ and set

$$h_1 = \sum_{n=1}^{\infty} \frac{c}{\sqrt{n} \log(n+1)} e_n,$$

where c is a normalizing constant such that $\|h_1\| = 1$. Expand h_1 to an orthonormal basis $\{h_n\}$. Now define an operator T on H by

$$Tx = \sum_{n=1}^{\infty} \frac{1}{2^n} \langle x, h_n \rangle e_n, \quad T^*x = \sum_{n=1}^{\infty} \frac{1}{2^n} \langle x, e_n \rangle h_n.$$

It is easy to show that $T \in S_p$ for every $p > 0$, but for $0 < p < 2$ we have

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T e_n, e_k \rangle|^p = \sum_{n=1}^{\infty} \frac{1}{2^{np}} \sum_{k=1}^{\infty} |\langle h_n, e_k \rangle|^p = \infty,$$

because in this case we have

$$\sum_{k=1}^{\infty} |\langle h_1, e_k \rangle|^p = c^p \sum_{k=1}^{\infty} \frac{1}{[\sqrt{k} \log(k+1)]^p} = \infty.$$

It is now natural for us to ask whether the “converse” of the theorems above is true. We show that the answer is no when p is not the upper end-point. The end-point case will be discussed in the next section. The next three propositions only require the operator T to be bounded, not necessarily in S_p .

Proposition 19. *Let $0 < p < 1$ and let T be any nonzero operator on H . Then there exists a frame $\{f_n\}$ such that $\{\langle T f_n, f_n \rangle\} \notin \ell^p$.*

Proof. Since $T \neq 0$, there exists a unit vector h such that $\langle Th, h \rangle \neq 0$. Fix such a vector h and set

$$e'_n = \frac{h}{\sqrt{n} \log(n+1)}, \quad n \geq 1.$$

For any $f \in H$ we have

$$\sum_{n=1}^{\infty} |\langle f, e'_n \rangle|^2 = \sum_{n=1}^{\infty} \frac{|\langle f, h \rangle|^2}{n[\log(n+1)]^2} \leq \|f\|^2 \sum_{n=1}^{\infty} \frac{1}{n[\log(n+1)]^2}.$$

Let $\{e_n\}$ be any orthonormal basis for H and let $\{f_n\} = \{e_n\} \cup \{e'_n\}$. Since the last series above converges and

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle f, e'_n \rangle|^2 = \|f\|^2 + \sum_{n=1}^{\infty} |\langle f, e'_n \rangle|^2,$$

we conclude that $\{f_n\}$ is a frame for H .

On the other hand, the sequence $\{\langle T f_n, f_n \rangle\}$ contains the subsequence $\{\langle T e'_n, e'_n \rangle\}$, which is not in ℓ^p for $0 < p < 1$. In fact,

$$\langle T e'_n, e'_n \rangle = \frac{\langle Th, h \rangle}{n[\log(n+1)]^2}$$

for all $n \geq 1$, which clearly shows that $\{\langle T e'_n, e'_n \rangle\} \notin \ell^p$ for $0 < p < 1$. This shows that $\{\langle T f_n, f_n \rangle\}$ is not in ℓ^p and completes the proof of the proposition. \square

Proposition 20. *Let $0 < p < 2$ and let T be any nonzero operator on H . There exists a frame $\{g_n\}$ such that $\{\|T g_n\|\} \notin \ell^p$.*

Proof. Consider the operator $S = T^*T$ and use Proposition 19 to find a frame $\{f_n\}$ such that $\{\langle Sf_n, f_n \rangle\}$ is not in $\ell^{p/2}$. This is clearly the same as $\{\|Tf_n\|\} \notin \ell^p$. \square

Proposition 21. *For any $0 < p < 2$ and any nonzero operator T on H there exists a frame $\{f_n\}$ such that*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle Tf_n, f_k \rangle|^p = \infty.$$

Proof. This follows from Proposition 20 and part (b) of Lemma 11. \square

We can also obtain versions of these propositions in terms of orthonormal bases. Note that the operators T obtained in the next three propositions are in S_p , not just bounded.

Proposition 22. *Suppose $0 < p < 1$ and $\{e_n\}$ is any orthonormal basis for H . Then there exists a positive operator $S \in S_p$ such that $\{\langle Se_n, e_n \rangle\} \notin \ell^p$.*

Proof. Fix a nonzero operator $T \in S_p$ and use Proposition 19 to find a frame $\{f_n\}$ such that $\{\langle Tf_n, f_n \rangle\} \notin \ell^p$. Let A denote the operator from Lemma 1 and consider the operator $S = A^*TA$. Then S is a positive operator in S_p and $\{\langle Se_n, e_n \rangle\} = \{\langle Tf_n, f_n \rangle\} \notin \ell^p$. \square

Proposition 23. *Suppose $0 < p < 2$ and $\{e_n\}$ is any orthonormal basis for H . Then there exists a positive operator $S \in S_p$ such that $\{\|Se_n\|\} \notin \ell^p$.*

Proof. By Proposition 22, there exists a positive operator $T \in S_{p/2}$ such that $\{\langle Te_n, e_n \rangle\} \notin \ell^{p/2}$. Let $S = \sqrt{T}$. Then S is a positive operator in S_p , $\|Se_n\|^2 = \langle Te_n, e_n \rangle$, and $\{\|Se_n\|\} \notin \ell^p$. \square

Proposition 24. *For any $0 < p < 2$ and any orthonormal basis $\{e_n\}$ there exists a positive operator $S \in S_p$ such that*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle Se_n, e_k \rangle|^p = \infty.$$

Proof. This follows from Proposition 23 and part (b) of Lemma 11. Note that the operator constructed right before Proposition 19 is not positive. \square

4. THE TRACE CLASS AND HILBERT-SCHMIDT CLASS

In this section we focus on two special classes of operators: the trace class S_1 and the Hilbert-Schmidt class S_2 . It is well known that a bounded linear operator T on H is in S_2 if and only if $\sum \|Te_n\|^2 < \infty$, where $\{e_n\}$

is any given orthonormal basis for H . Also, for $T \geq 0$, $T \in S_1$ if and only if $\sum \langle T e_n, e_n \rangle < \infty$. We show that these results remain true when the orthonormal basis $\{e_n\}$ is replaced by a frame.

Theorem 25. *Suppose T is a positive operator on H . Then the following conditions are equivalent.*

- (a) T is in the trace class S_1 .
- (b) $\{\langle T f_n, f_n \rangle\} \in \ell^1$ for every frame $\{f_n\}$.
- (c) $\{\langle T f_n, f_n \rangle\} \in \ell^1$ for some frame $\{f_n\}$.

Proof. This follows from Theorems 7 and 16. □

An equivalent version of Theorem 25 above is the following.

Theorem 26. *Let T be a bounded linear operator on H . Then the following conditions are equivalent.*

- (a) T is in the Hilbert-Schmidt class S_2 .
- (b) $\{\|T f_n\|\} \in \ell^2$ for every frame $\{f_n\}$.
- (c) $\{\|T f_n\|\} \in \ell^2$ for some frame $\{f_n\}$.

Proof. Note that T is Hilbert-Schmidt if and only if T^*T is trace class. Since $\langle T^*T f_n, f_n \rangle = \|T f_n\|^2$, the desired result follows from Theorem 25. Alternatively, the desired result follows from Theorems 2 and 17. □

When $\{f_n\}$ is a frame, it is clear that the condition $\{\|T f_n\|\} \in \ell^2$ is equivalent to the condition

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle T f_n, f_k \rangle|^2 < \infty.$$

Therefore, conditions (b) and (c) in Theorem 26 above can also be stated in terms of the double-indexed sequence $\{\langle T f_n, f_k \rangle\}$.

5. AN APPLICATION

In this section, we consider a special class of frames in the Bergman space of the unit disk, namely, normalized reproducing kernels induced by sampling sequences. We use this to obtain an integral condition for a bounded linear operator on the Bergman space to belong to the Schatten class S_p .

Thus we let A^2 denote the space of analytic functions f in the unit disk \mathbb{D} such that

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where dA is area measure on \mathbb{D} normalized so that \mathbb{D} has area 1. As a closed subspace of $L^2(\mathbb{D}, dA)$, A^2 is a Hilbert space. In fact, A^2 is a reproducing Hilbert space whose reproducing kernel is the well-known Bergman kernel

$$K_w(z) = K(z, w) = \frac{1}{(1 - z\bar{w})^2}.$$

For any $w \in \mathbb{D}$ let k_w denote the function in A^2 defined by

$$k_w(z) = \frac{K(z, w)}{\sqrt{K(w, w)}} = \frac{1 - |w|^2}{(1 - z\bar{w})^2}.$$

Each k_w is a unit vector in A^2 , called the normalized reproducing kernel at w .

A sequence $\{w_n\}$ in \mathbb{D} is called a sampling sequence for the Bergman space A^2 if there exists a positive constant C such that

$$C^{-1}\|f\|^2 \leq \sum_{n=1}^{\infty} (1 - |w_n|^2)^2 |f(w_n)|^2 \leq C\|f\|^2$$

for all $f \in A^2$. This condition can be written as

$$C^{-1}\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, k_{w_n} \rangle|^2 \leq C\|f\|^2.$$

Therefore, $\{w_n\}$ is a sampling sequence for the Bergman space if and only if the sequence $\{k_{w_n}\}$ is a frame in A^2 . See [5] for the theory of Bergman spaces, including the notions of normalized reproducing kernels and sampling sequences. Sampling sequences for the Bergman space are characterized in [7].

Some results obtained in the paper can be stated in terms of sampling sequences. As one particular example, we infer from Theorem 26 that if $\{w_n\}$ is a sampling sequence for the Bergman space, then a bounded linear operator T on A^2 belongs to the Hilbert-Schmidt class S_2 if and only if it satisfies the condition

$$\sum_{n=1}^{\infty} \|Tk_{w_n}\|^2 < \infty.$$

Equivalently, a positive operator T on A^2 belongs to the trace class if and only if

$$\sum_{n=1}^{\infty} \langle Tk_{w_n}, k_{w_n} \rangle < \infty.$$

Lemma 27. *Suppose T is a bounded linear operator on A^2 and $0 < p < \infty$. Then the function $F(w) = \|TK_w\|^p$ is subharmonic in \mathbb{D} .*

Proof. Without loss of generality we assume that $T \neq 0$. It is then clear that the function F has isolated zeros in \mathbb{D} . Furthermore, away from the zeros of F , it follows from

$$F(w) = \langle TK_w, TK_w \rangle^{\frac{p}{2}}$$

that

$$\frac{\partial F}{\partial \bar{w}} = \frac{p}{2} \langle TK_w, TK_w \rangle^{\frac{p}{2}-1} \langle TK_w^*, K_w \rangle,$$

where

$$K_w^*(z) = \frac{\partial}{\partial \bar{w}} \frac{1}{(1 - z\bar{w})^2} = \frac{2z}{(1 - z\bar{w})^3}.$$

Differentiating one more time, we obtain

$$\begin{aligned} \frac{\partial^2 F}{\partial w \partial \bar{w}} &= \frac{p}{2} \left(\frac{p}{2} - 1 \right) \langle TK_w, TK_w \rangle^{\frac{p}{2}-2} \langle TK_w^*, K_w \rangle \langle TK_w, TK_w^* \rangle \\ &\quad + \frac{p}{2} \langle TK_w, T_w \rangle^{\frac{p}{2}-1} \langle TK_w^*, TK_w^* \rangle \\ &= \frac{p}{2} \left[\frac{p}{2} - 1 \right] \|TK_w\|^{p-4} |\langle TK_w^*, TK_w \rangle|^2 + \frac{p}{2} \|TK_w\|^{p-2} \|TK_w^*\|^2 \\ &= \frac{p}{2} \|TK_w\|^{p-4} \left[\left(\frac{p}{2} - 1 \right) |\langle TK_w^*, TK_w \rangle|^2 + \|TK_w\|^2 \|TK_w^*\|^2 \right] \\ &\geq \frac{p}{2} \|TK_w\|^{p-4} [\|TK_w\|^2 \|TK_w^*\|^2 - |\langle TK_w^*, TK_w \rangle|^2] \\ &= \frac{p}{2} \|TK_w\|^{p-4} [\|TK_w\| \|TK_w^*\| + |\langle TK_w^*, TK_w \rangle|] \cdot \\ &\quad \cdot [\|TK_w\| \|TK_w^*\| - |\langle TK_w^*, TK_w \rangle|] \\ &\geq 0. \end{aligned}$$

The last inequality above is a consequence of the Cauchy-Schwarz inequality. \square

Theorem 28. Suppose T is a bounded linear operator on A^2 and

$$d\lambda(w) = \frac{1}{(1 - |w|^2)^2} dA(w)$$

is the Möbius invariant area measure on \mathbb{D} . Then the condition

$$\int_{\mathbb{D}} \|Tk_w\|^p d\lambda(w) < \infty$$

is sufficient for $T \in S_p$ when $0 < p \leq 2$ and it is necessary for $T \in S_p$ when $2 \leq p < \infty$. Consequently, T is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \|Tk_w\|^2 d\lambda(w) = \int_{\mathbb{D}} \|TK_w\|^2 dA(w) < \infty.$$

Proof. The result follows from Theorem 6.6 in [9], because $T \in S_p$ if and only if the positive operator $S = T^*T$ is in $S_{p/2}$,

$$\tilde{S}(w) = \langle Sk_w, k_w \rangle = \|Tk_w\|^2$$

for all $w \in \mathbb{D}$, and $0 < p \leq 2$ if and only if $0 < p/2 \leq 1$.

The proof of Theorem 6.6 in [9] depends on the notion of the Berezin transform and the spectral decomposition for positive operators. Here we give an independent proof in the case $0 < p \leq 2$ that is based on sampling sequences and subharmonicity.

Fix a sampling sequence $\{w_n\} \subset \mathbb{D}$ for the Bergman space A^2 such that $\{w_n\}$ is separated in the Bergman metric β , say, $\beta(w_i, w_j) > 2r$ for some positive number r and all $i \neq j$. See [5] for the existence of such a sequence. Let $D(w_n, r) = \{z \in \mathbb{D} : \beta(z, w_n) < r\}$ denote the Bergman metric ball centered at w_n with radius r .

By Lemma 27, the function $w \mapsto \|TK_w\|^p$ is subharmonic. It follows from the proof of Proposition 4.13 in [9] that there exists a positive constant C , independent of n and T , such that

$$\|TK_{w_n}\|^p \leq \frac{C}{|D(w_n, r)|} \int_{D(w_n, r)} \|TK_w\|^p dA(w)$$

for all $n \geq 1$, where $|D(w_n, r)|$ is the area of $D(w_n, r)$. By Proposition 4.5 in [9] and the remarks following it, we have

$$|D(w_n, r)| \sim (1 - |w_n|^2)^2 \sim (1 - |w|^2)^2$$

for $w \in D(w_n, r)$. It follows that there exists another positive constant C , independent of n and T , such that

$$\|Tk_{w_n}\|^p \leq C \int_{D(w_n, r)} \|Tk_w\|^p d\lambda(w)$$

for all $n \geq 1$. Since the Bergman metric balls $D(w_n, r)$ are mutually disjoint, we have

$$\sum_{n=1}^{\infty} \|Tk_{w_n}\|^p \leq C \int_{\mathbb{D}} \|Tk_w\|^p d\lambda(w).$$

The desired result now follows from Theorem 17. □

The ideas and results of this section clearly generalize to many other reproducing kernel Hilbert spaces, including weighted Bergman spaces on various domains and Fock spaces on \mathbb{C}^n .

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